“Perfect” Power Law Graphs: Generation, Sampling, Construction, and Fitting

Jeremy Kepner

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Outline

- Introduction
- Sampling
- Sub-sampling
- Reuter’s Data
- Summary
Goals

• Develop a background model for graphs based on “perfect” power law

• Examine effects of sampling such a power law

• Develop techniques for comparing real data with a power law model
Detection Theory

DETECTION OF SIGNAL IN NOISE

ASSUMPTIONS
- Background (noise) statistics
- Foreground (signal) statistics
- Foreground/background separation
- Model \( \approx \) reality

DETECTION OF SUBGRAPHS IN GRAPHS

Example subgraph of interest: Fully connected (complete)

Example background model: Powerlaw graph

Can we construct a background model based on power law degree distribution?
"Perfect" Power Law Matrix Definition

- Graph represented as a rectangular sparse matrix
  - Can be undirected, multi-edged, self-loops, disconnected, hyper edges, …
- Out/in degree distributions are independent first order statistics
  - Only constraint: $\sum n(d_{out}) d_{out} = \sum n(d_{in}) d_{in} = M$
Power Law Distribution Construction

- **Perfect power law matlab code**

```matlab
function [di ni] = PPL(alpha,dmax,Nd)
logdi = (0:Nd) * log(dmax) / Nd;
di    = unique(round(exp(logdi)));
logni = alpha * (log(dmax) - log(di));
ni    = round(exp(logni));
```

- **Parameters**
  - $\alpha$ = slope
  - $d_{max}$ = largest degree vertex
  - Nd = number of bins (before unique)

- **Simple algorithm naturally generates perfect power law**
- **Smooth transition from integer to logarithmic bins**
- **“Poor man’s” slope estimator:** $\alpha = \log(n_1)/\log(d_{max})$
Power Law Edge Construction

- Power law vertex list matlab code

```matlab
function v = PowerLawEdges(di,ni);
A1 = sparse(1:numel(di),ni,di);
A2 = fliplr(cumsum(fliplr(A1),2));
[tmp tmp d] = find(A2);
A3 = sparse(1:numel(d),d,1);
A4 = fliplr(cumsum(fliplr(A3),2));
[v tmp tmp] = find(A4);
```

- Degree distribution independent of
  - Vertex labels
  - Edge pairing
  - Edge order

- Algorithm generates list of vertices corresponding to any distribution
- All other aspects of graph can be set based on desired properties
Fitting $\alpha$, $N$, $M$

- Power law model works for any $\alpha > 0$, $d_{\text{max}} > 1$, $N_d > 1$

- Desire distribution that fits $\alpha$, $N$, $M$

- Can invert formulas
  - $N = \sum_i n(d_i)$
  - $M = \sum_i n(d_i) d_i$

- Highly non-linear; requires a combination of
  - Exhaustive search, simulated annealing, and Broyden’s algorithm

- Given $\alpha$, $N$, $M$ can solve for $N_d$ and $d_{\text{max}}$
- Not all combinations of $\alpha$, $N$, $M$ are consistent with power law
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Graph Construction Effects

• Generate a perfect power law NxN randomize adjacency matrix A
  – $\alpha = 1.3$, $d_{\text{max}} = 1000$, $N_d = 50$
  – $N = 18K$, $M = 84K$

• Make undirected, unweighted, with no self-loops
  
  $A = \text{triu}(A + A')$;
  $A = \text{double}(\text{logical}(A))$;
  $A = A - \text{diag}(\text{diag}(A))$;

• Graph theory best for undirected, unweighted graphs with no self-loops
• Often “clean up” real data to apply graph theory results
• Process mimics “bent broom” distribution seen in real data sets
Power Law Recovery

Procedure

• Compute $\alpha$, N, M from measured

• Fit perfect power law to these parameters

• Rebin measured data using perfect power law degree bins

• Perfect power law fit to “cleaned up” graph can recover much of the shape of the original distribution
Correlation Construction Effects

• Generate a perfect power law \( N \times N \) randomize incidence matrix \( E \)
  – \( \alpha = 1.3, \ d_{\text{max}} = 1000, \ N_d = 50 \)
  – \( N = 18K, \ M = 84K \)

• Make unweighted and use to form correlation matrix \( A \) with no self-loops

\[
E = \text{double}(\text{logical}(E));
A = \text{triu}(E' \ast E);
A = A - \text{diag}(	ext{diag}(A));
\]

• Correlation graph construction from incidence matrix results in a “bent broom” distribution that strongly resembles a power law
**Power Law Lost**

**Procedure**

- Compute $\alpha$, $N$, $M$ from measured data.
- Fit perfect power law to these parameters.
- Rebin measured data using perfect power law degree bins.

- Perfect power law fit to correlation shows non-power law shape.
- Reveals “witches nose” distribution.
Power Law Preserved

- In degree is power law
  \[ \alpha = 1.3, \, d_{\text{max}} = 1000, \, N_d = 50 \]
  - \( N = 18K, \, M = 84K \)
- Out degree is constant
  - \( N = 16K, \, M = 84K \)
  - Edges/row = 5 (exactly)

- Make unweighted and use to form correlation matrix \( A \) with no self-loops

- Uniform distribution on correlated dimension preserves power law shape
Edge Ordering: Densification

- Compute $M/N$ cumulatively and piecewise for 2 orderings
  - Linear
  - Random

- By definition $M/N$ goes from 1 to infinity for finite $N$

- Elimination of multi-edges reduces $M$ and causes $M/N$ to grow more slowly

- “Densification” is the observation that $M/N$ increases with $N$
- Densification is a natural byproduct of randomly drawing edges from a power law distribution
- Linear ordering has constant $M/N$
Edge Ordering: Power Law Exponent ($\alpha$)

- Compute $\alpha$ cumulatively and piecewise for 2 orderings
  - Linear
  - Random

- Edge ordering and sampling have large effect on the power law exponent

- Power law exponent is fundamental to distribution
  - Strongly dependent on edge ordering and sample size
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Sub-Sampling Challenge

- Anomaly detection requires good estimates of background

- Traversing entire data sets to compute background counts is increasingly prohibitive
  - Can be done at ingest, but often is not

- Can background be accurately estimated from a sub-sample of the entire data set?
Sampling a Power Law

- Generate power law
- Select fraction of edges
Linear Degree Estimate

- Divide measured degree by fraction
- Accurate for high degree
- Overestimates low degree
- Can we do better?

Whole distribution

Linear estimate
Non-Linear Degree Estimate

- Assume power law input
- Create non-linear estimate
- Matches median degree

Whole distribution

Non-Linear estimate
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Reuter’s Incidence Matrix

- Entities extracted from Reuter’s Corpus
- \( E(i,j) = \# \) times entity appeared in document
- \( N_{\text{doc}} = 797677 \)
- \( N_{\text{ent}} = 47576 \)
- \( M = 6132286 \)
- Four entity classes with different statistics
  - LOCATION
  - ORGANIZATION
  - PERSON
  - TIME
- Fit power law model to each entity class
E(:,PERSON) Degree Distribution

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<th>N</th>
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<th>α</th>
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Procedure

- Make unweighted and use to form correlation matrix A with no self-loops

\[
E = \text{double}(\text{logical}(E));
\]
\[
A = \text{triu}(E' \times E);
\]
\[
A = A - \text{diag}(\text{diag}(A));
\]

- Perfect power law fit to correlation shows non-power law shape
- Reveals “witches nose” distribution
Document Densification

- Constant M/N consistent with sequential ordering of documents
Entity Densification

- Increasing M/N consistent with random ordering of entities
Document Power Law Exponent ($\alpha$)

- Increasing $\alpha$ consistent with sequential ordering of documents
Entity Power Law Exponent ($\alpha$)

- Decreasing $\alpha$ consistent with random ordering of entities
Summary

- Developed a background model for graphs based on “perfect” power law
  - Can be done via simple heuristic
  - Reproduces much of observed phenomena

- Examine effects of sampling such a power law
  - Lossy, non-linear transformation of graph construction mirrors many observed phenomena

- Traditional sampling approaches significantly overestimate the probability of low degree vertices
  - Assuming a power law distribution it is possible to construct a simple non-linear estimate that is more accurate

- Develop techniques for comparing real data with a power law model
  - Can fit perfect power-law to observed data
  - Provided binning for statistical tests
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Appendix
Sub-Sampling Formula

- $f =$ fraction of total edges sampled
- $n_1 =$ # of vertices of degree 1
- $d_{\text{max}} =$ maximum degree
- Allowed slope: $\ln(n_1)/\ln(d_{\text{max}}/f) < \alpha < \ln(n_1)/\ln(d_{\text{max}})$

- Cumulative distribution
  \[ P(\alpha, d) = (f^{1-\alpha} d_{\text{max}}^\alpha / n_1) \sum_{i<d} i^{1-\alpha} e^{-fi} \]

- Find $\alpha^*$ such that $P(\alpha^*, \infty) = 1$
- Find $d_{50\%}$ such that $P(\alpha^*, d_{50\%}) = 1/2$
- Compute $K = 1/(1 + \ln(d_{50\%})/\ln(f))$

- Non-linear estimate of true degree of vertex $v$ from sample $d(v)$
  \[ d(v) = d(v) / f^{1-1/(K d(v))} \]